Solution to Home work 1

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1. Sample space for die experiment $\Omega = \{1, 2, 3, 4, 5, 6\}$
2. Event space: Possible events

$$\begin{aligned} E = \{&\{\Phi\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}\ , \\ &\{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}, \\ &\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\} \ldots \{1,2,3,4,5,6\}\} \end{aligned} \tag{1}$$

One can easily verify that the fundemental events are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$, all other events can be formed by taking unions and intersection of these events. In this way we have characterized Ω and E for this experiment. Now we know that probability measure for each event is

$$Pr(\{i\}) = 1/6, \ i = 1, \ldots, 6 \tag{2}$$

And for any two events

$$Pr(\{1\} \cup \{2\}) = 1/6 + 1/6 = 1/3 \tag{3}$$

Hence the probability triplet is specified.

2. For $f(x)$ to be valid *pdf*, we know that

$$\begin{aligned} 1 &= \int_0^1 ce^{-x} dx \\ &= c\left(1 - e^{-1}\right) \end{aligned} \tag{4}$$

hence

$$c = \frac{1}{1 - e^{-1}} = 1.582$$

3- This problem seems to be trivial, but it is not. Actually $X + Y$ is now a joint random variable with joint distribution $P(X, Y)$. But we assume that X and Y are statistically independent, hence $P(X, Y) = P(X)P(Y)$. Using this fact we now proceed as follows.
By definition of expectation

$$\begin{aligned} \mathbf{E}(X + Y) &= \sum_i \sum_j (x_i + y_j)\, p(x_i, y_j) \\ &= \sum_i \sum_j x_i p(x_i, y_j) + \sum_j y_j p(x_i, y_j) \\ &= \sum_i \sum_j x_i p(x_i) p(y_j) + \sum_i \sum_j y_j p(x_i) p(y_j) \\ &= \sum_i x_i p(x_i) \sum_j p(y_j) + \sum_j y_j p(y_j) \sum_i p(x_i) \\ &= \sum_i x_i p(x_i) + \sum_j y_j p(y_j) \\ &= \mathbf{E}(X) + \mathbf{E}(Y) \end{aligned} \tag{5}$$

4- Average information is measured in terms of entropy of the random variable, which is

$$
\begin{aligned}
H(X) &= -\sum_{i=1}^{2} p(x_i)\log_2(p(x_i)) \\
&= -p(x_1)\log_2(p(x_1)) - p(x_2)\log_2(p(x_2)) \\
&= -1/2\log_2(1/2) - 1/2\log_2(1/2) = 1 \qquad (6)
\end{aligned}
$$

hence average information is 1 bits/sec.